

Coloured Koszul duality and strongly homotopy operads

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Abstract

This paper proves Koszul duality for coloured operads and uses it to introduce strongly homotopy operads as a suitable homotopy invariant version of operads. It shows that \mathbb{Q} -chains on configuration spaces of points in the unit disk form a strongly homotopy operad quasi isomorphic to the chains on the little disks operad.

1 Introduction

Throughout this paper operads are operads in the category of dg vector spaces over a field k of characteristic 0.

In some situations the notion of operad is too restrictive. Think of the following.

- (i). Given two quasi isomorphic operads P and Q there need not exist a quasi isomorphism $P \rightarrow Q$ of operads.
- (ii). Given an operad P , one usually can not transfer a strongly homotopy P -algebra structure from a dg vector space W to a dg vector space V using a map $\text{End}_W \rightarrow \text{End}_V$, which from the operadic point of view would be the most natural thing to try.
- (iii). The singular k -chains on configuration spaces of distinct ordered points in the unit disk do not form an operad quasi isomorphic to the k -chains on the little disks operad in any straightforward manner, unless one uses Fulton-MacPherson compactification.

The way in which this paper deals with these difficulties is by defining a somewhat weaker version of operads, strongly homotopy operads and morphisms between them. The definition of a strongly homotopy operad is based on the analogy between operads and associative algebras advocated by Ginzburg-Kapranov [3]. In this analogy strongly homotopy operads correspond to A_∞ -algebras (i.e. strongly homotopy associative algebras). This paper shows that one can make the analogy very precise using Koszul duality for the \mathbb{N} -coloured operad which as algebras non-symmetric pseudo operads. In fact one recovers the associative algebra analogon when restricting to s.h. operads P that as collections are concentrated in $P(1)$.

The main results can be summarized as follows.

- (i). Every quasi isomorphism of strongly homotopy operads admits a quasi inverse. Consequently, two augmented operads P and Q are quasi isomorphic iff there exists a quasi isomorphism $P \rightsquigarrow Q$ of strongly homotopy operads.
- (ii). If W and V are two dg vector spaces and $i : V \longrightarrow W$, $r : W \longrightarrow V$ and $H : W \longrightarrow W[1]$ are dg maps such that H is a chain homotopy between $i \circ r$ and the identity on W , then there exists a morphism of strongly homotopy operads

$$\mathrm{End}_W \rightsquigarrow \mathrm{End}_V,$$

given by an explicit formula. This map is a quasi isomorphism if r and i are quasi isomorphisms. If P is an operad, and W is a strongly homotopy P -algebra, this map can be used to transport the strongly homotopy P -algebra structure to V .

- (iii). The k -chains on configuration spaces of ordered distinct points in the unit disk form a strongly homotopy operad quasi isomorphic to the k -chains on the little disks operad.

Further applications of strongly homotopy operads related to formal deformation theory for operads and their algebras, and L_∞ -algebras can be found in my thesis [10].

Plan of the paper

The preliminaries (Section 2) fix some notation. Section 3 briefly introduces coloured operads, and then shows Koszul duality can be extended to coloured operads. Section 4 applies this to the \mathbb{N} -coloured operad PsOpd which has as algebras non-symmetric pseudo operads, and gives an equivariant version of strongly homotopy PsOpd -algebras that defines strongly homotopy operads. Finally, it considers morphisms of strongly homotopy operads and proves the first main result. Section 5 proves the second main result, and considers its implications for strongly homotopy P -algebras. The application of these results to the operad PsOpd lead to the proof of the third main result.

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2 Preliminaries

I work in the category of dg vector spaces over a field k of characteristic 0. If V is a dg vector space, and $v \in V$ is an homogeneous element, then its degree will be denoted by $|v|$. I use the cohomological convention: the differential d of the dg vector space V is a map of degree $+1$. Let $V^n = \{v \in V \mid |v| = n\}$ be the space of homogeneous elements of degree n . Then $V[m]$ is the dg vector space with $(V[m])^n = V^{n-m}$. Later on I might be a bit sloppy and leave out the ‘dg’ since I only work with differentially graded objects. Let V and W be (dg) vector spaces. Recall

that the symmetry τ of the tensor product involves the natural signs $\tau : v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ on homogeneous elements. I use the Koszul convention $f \otimes g(x \otimes y) := (-1)^{|g||x|} f(x) \otimes g(y)$ for homogeneous maps $f : V \rightarrow V'$ and $g : W \rightarrow W'$ applied to homogeneous elements $x \in V$ and $y \in W$. In combination with the shift $[m]$ this reduces the number of signs significantly.

By S_n we denote the symmetric group on n letters, and by kS_n its group algebra which is the vector space spanned by the set S_n whose multiplication is the linear extension of multiplication in S_n . If a group G acts on a vector space V , the coinvariants of the group action are denoted V_G and the invariants by V^G .

Operads

A non-symmetric operad is a sequence $\{P(n)\}_{n \geq 1}$ of (dg) vector spaces together with composition maps

$$\gamma : P(n) \otimes P(m_1) \otimes \dots \otimes P(m_n) \rightarrow P(m_1 + \dots + m_n),$$

and an identity element $\text{id} \in P(1)$. These structures satisfy the usual associativity and identity axioms (cf. Getzler-Jones [2], and Markl-Shnider-Stasheff [8]).

A collection P is a sequence of vector spaces $\{P(n)\}_{n \geq 1}$ such that each $P(n)$ has a right S_n -module structure. An (symmetric) operad is a collection P together with a non-symmetric operad structure on the sequence of vector spaces, and composition is equivariant with respect to the S_n -actions in the usual sense (cf. Getzler-Jones [2], and Markl-Shnider-Stasheff [8]). Dually (in the sense of inverting direction of arrows in the defining diagrams), one defines (non-symmetric) cooperads.

(Non-symmetric) pseudo operads, are the non-unital analogon of (non-symmetric) operads. A non-symmetric pseudo operad P is a sequence of dg vector spaces P together with dg maps $\circ_i : P(n) \otimes P(m) \rightarrow P(m+n-1)$ for $i = 1, \dots, n$, which satisfy the appropriate associativity conditions. A non-symmetric operad gives rise to a non-symmetric pseudo operad by

$$p \circ_i q = \gamma(p; \text{id}^{i-1}, q, \text{id}^{n-i}) \quad (2.1)$$

for $p \in P(n)$ and $q \in P(m)$. Pseudo operads are the equivariant version of this, starting from a collection P . The category of (non-symmetric) pseudo operads is equivalent to the category of augmented operads. That is, operads P such that the inclusion of the identity is split as a map of operads. Throughout this paper I assume all operads except endomorphism operads to be augmented.

A graph η consist of sets $\mathbf{v}(\eta)$ of vertices, a set $\mathbf{e}(\eta)$ of internal edges, and a set $\mathbf{l}(\eta)$ of external edges or legs; together with a map that assigns to each edge a pair of (not necessary distinct) vertices and a map that assigns to each leg a vertex. To draw a graph, draw a dot for each vertex v , and for each edge e draw a line between the two vertices assigned to it, and for each leg draw a line that in one end ends in the vertex assigned to it. If $v \in \mathbf{v}(\eta)$, denote by $\mathbf{l}(v) \subset \mathbf{e}(v) \cup \mathbf{l}(\eta)$ the set of legs and edges attached to v and call elements of $\mathbf{l}(v)$ the legs of v . A morphism of graphs consists of morphisms of vertices, edges, and legs compatible with the structure maps.

A connected graph t is a tree if $|\mathbf{v}(t)| = |\mathbf{e}(t)| + 1$. A rooted tree is a tree together with a basepoint $r \in \mathbf{l}(t)$, the root, and together with a

bijection $\mathbf{l}(t) \longrightarrow \{0, \dots, n_t\}$ that sends the root to 0, where $n_t = |\mathbf{l}(t)| - 1$. In a rooted tree t , each of the sets $\mathbf{l}(v)$ has a natural basepoint, the leg in the direction of the root. A planar tree is a rooted tree together with for each $v \in \mathbf{v}(t)$ a bijection $\mathbf{l}(t) \longrightarrow \{0, \dots, n_v\}$ that sends the basepoint to 0, for $n_v := |\mathbf{l}(v)| - 1$. For any planar tree t define

$$C(t) := \bigotimes_{v \in \mathbf{v}(t)} C(n_v).$$

The free pseudo operad TC and the ‘cofree’ pseudo cooperad $T'C$ on a collection C satisfy

$$TC(n) = \operatorname{colim}_t C(t), \quad T'C(n) = \lim_t C(t),$$

where both limit and colimit are over the groupoid of planar trees with n external edges different from the root with isomorphisms of rooted trees as maps. These maps need not preserve the planar structure, but do preserve the labeling of the legs different from the root in $\mathbf{l}(t)$ by $1, \dots, n$. The operad structure on TC is given by grafting trees, while the cooperad structure on $T'C$ is given by cutting edges.

It is useful to be a bit more explicit on the arrows of the diagram over which we take the (co)limit in defining TC and $T'C$. Let $\sigma : t \mapsto t'$ be an isomorphism of rooted trees. For $v \in \mathbf{v}(t)$ and $v' \in \mathbf{v}(t')$, if $\sigma(v) = v'$ it induces $C(n_v) \longrightarrow C(n_{v'})$. Define

$$C(\sigma) : C(t) = \bigotimes_{v \in t} C(n_v) \longrightarrow \bigotimes_{v' \in t'} C(n_{v'}) = C(t'),$$

as the tensor product over $v \in \mathbf{v}(t)$ of these maps. Note that σ restricts to a bijection $\sigma|_{\mathbf{l}(t)} : \mathbf{l}(t) \longrightarrow \mathbf{l}(t')$, and that σ being an isomorphism of rooted trees implies compatibility of the labeling of the external edges of the trees with $\sigma|_{\mathbf{l}(t)}$.

If P is an operad, then there is a natural differential ∂_P on $T'(P[-1])$. That is, ∂_P is a square-zero coderivation of degree $+1$. The resulting cooperad $BP = (T'(P[-1]), \partial)$ is the bar construction on P . For more extensive background on this and on (co)operads in general read Ginzburg-Kapranov [3], Getzler-Jones [2], and Markl-Shnider-Stasheff [8].

3 Coloured Koszul duality

Coloured operads

Denote by \mathbf{n} the set $\{0, 1, \dots, n\}$ for $n \geq 0$, and let I be a set. An *I-coloured collection* (or *I-collection*) P is a set $\{P(n, i)\}_{(n, i: \mathbf{n} \rightarrow I)}$ of dg vector spaces indexed by the sets $\mathbf{n} = \{0, 1, \dots, n\}$ for all $n \geq 1$, and by all surjections $\mathbf{n} \longrightarrow I$; together with a right S_n -action on $\bigoplus_{i: \mathbf{n} \rightarrow I} P(n, i)$ such that for $\sigma \in S_n$ the action satisfies $(P(n, i))\sigma \subset P(n, i\sigma)$, where $i\sigma : \mathbf{n} \longrightarrow I$ is i precomposed by the permutation σ applied to $\{1, \dots, n\} \subset \mathbf{n}$. The values of i are called labels. More particular, $i(0)$ is the output label, and $i(1), \dots, i(n)$ are the labels of the inputs $1, \dots, n$.

An *I-coloured pseudo operad* (or *I-pseudo operad*) is an *I-collection* P , together with compositions

$$\circ_l : P(n, i) \otimes P(m, j) \longrightarrow P(m+n-1, i \circ_l j) \quad \text{for } l \leq n, \text{ s.t. } i(k) = j(0),$$

(compare equation (2.1)) where $i \circ_l j : (\mathbf{m} + \mathbf{n} - \mathbf{1}) \longrightarrow I$ satisfies

$$i \circ_l j(k) = \begin{cases} i(k) & \text{if } 0 \leq k < l \\ j(k - l + 1) & \text{if } l \leq k < l + m \\ i(k - m) & \text{if } k \geq l + m. \end{cases}$$

These data satisfy the compatibility relations for \circ_k -operations of a pseudo operad (associativity, equivariance) whenever these make sense.

An *I-coloured operad* (or *I-operad*) is an *I*-pseudo operad together with for each $\alpha \in I$ an identity $\text{id}_\alpha \in P(\mathbf{1}, \alpha)$, where $\alpha : \mathbf{1} \longrightarrow I$ is the constant map with value α . These identities act as units with respect to any well defined composition.

Similarly, define *I*-pseudo cooperads, and *I*-cooperads by inverting the arrows in the defining diagrams.

3.1 EXAMPLE There is an obvious 1-1 correspondence between operads and $*$ -operads, where $*$ is the one-point set.

Let I be a set. and Let $V = \{V_\alpha\}_{\alpha \in I}$ be a set of vector spaces. Denote by $\text{Hom}_k(-, -)$ the k -linear maps (the internal Hom functor). Then $\text{End}_V(n, i) := \text{Hom}_k(V_{i(1)} \otimes \dots \otimes V_{i(n)}, V_{i(0)})$ defines a *I*-operad with respect to the S_n -action on inputs and the obvious composition of maps where $\varphi \circ_l \psi$ uses the output of the map ψ as the l -th input of φ . This *I*-operad is called the *endomorphism operad* of V .

Let P be an *I*-operad. A *P-algebra* V is a set of vector spaces $V = \{V_\alpha\}_{\alpha \in I}$ together with a morphism of *I*-operads $P \longrightarrow \text{End}_V$.

Let I be a set, and denote by A_I the (non-unital) associative algebra generated by generators $[\alpha]$ for $\alpha \in I$ with the multiplication $[\alpha] \cdot [\alpha'] = \delta_{\alpha\alpha'} \cdot [\alpha]$, where δ is the Kronecker delta on the set I . For an associative algebra A , recall the definition of an *A*-pseudo operad as a pseudo operad in the category of *A*-modules. The following is now quite straightforward.

3.2 EXAMPLE Every *I*-coloured collection P gives rise to a collection in the collection of A_I -modules if we interpret $\bigoplus_{i: \mathbf{n} \rightarrow I} P(n, i)$ as a decomposition in eigenspaces of the left A_I and right $A_I^{\otimes n}$ -action with eigenvalue 1. The left action of a generator $[\alpha]$ on $P(n, i)$ is (again in terms of the Kronecker delta) $\delta_{\alpha i(0)} \cdot \text{id}$ and the right action of $[\alpha_1] \otimes \dots \otimes [\alpha_n]$ on $P(n, i)$ is $\delta_{\alpha_1 i(1)} \cdot \dots \cdot \delta_{\alpha_n i(n)} \cdot \text{id}$.

3.3 PROPOSITION (MARKL [9]) *There is a 1-1 correspondence between I-pseudo operads and A_I -pseudo operads together with a decomposition*

$$P(n) = \bigoplus_{i: \mathbf{n} \rightarrow I} P(n, i)$$

that makes P an A_I -collection (with the action of Example 3.2 above) and such that for $p \in P(n, i)$ and $q \in P(m, j)$

$$p \circ_l q = 0 \quad \text{if } j(0) \neq i(l).$$

This correspondence describes I -operads as a full subcategory of A_I -operads.

Koszul duality for I -operads

I assume the reader is familiar with Koszul duality for operads as introduced in Ginzburg-Kapranov [3] and its description using cooperads in Getzler-Jones [2]. To prove that Koszul duality works for I -operads it suffices to show that I -operads are closed under the relevant constructions in the category of A_I -operads.

3.4 LEMMA *The bar construction B_{A_I} from A_I -pseudo operads to A_I -pseudo cooperads restricts to a functor B_I from I -operads to I -cooperads.*

PROOF Let P be an I -pseudo operad considered as an A_I -pseudo operad. Recall that $B_{A_I}P(n)$ decomposes as a sum over trees with n leaves with vertices labeled by elements of P . Each action of A_I corresponds to an input or output in $B_{A_I}P(n)$. It thus is the action on the label of the vertex to which the corresponding leaf or root is attached. We get a decomposition of $B_{A_I}P(n)$ by the generators $[\alpha]$ that do not vanish on these labels. QED

An ideal J of an I -operad P is a sub I -collection J of P such that $p \circ_I q \in J$ iff either p or q is an element of J . Denote the free I -operad on a collection E by $T_I E$. An I -operad is called quadratic if it is of the form $T_I E/R$, where $E(n, i) = 0$ if $n \neq 2$, and R is an ideal generated by elements in $\bigoplus_{i:3 \rightarrow I} T_I E(3, i)$. Quadratic operads are naturally augmented.

3.5 DEFINITION The *Koszul dual I -cooperad* P^\perp of an I -operad P is its Koszul dual as an A_I -operad. The Lemma below shows this is well defined. An quadratic I -operad is Koszul if $P^\perp \longrightarrow B_A(P)$ is a quasi isomorphism of A_I -cooperads. The Koszul dual I -operad is $P^! = (P^\perp)^* \otimes \Lambda$, the linear dual of P^\perp tensored with the determinant operad (cf. Getzler-Jones [2]).

3.6 LEMMA *If P is a quadratic I -operad, then the Koszul dual P^\perp of P is an I -cooperad.*

PROOF Let $P = T_I E/R$ be a quadratic I -operad. The free A_I -cooperad $T'(E[-1])$ is an I -cooperad by the same argument on trees as above. Moreover, by categorical generalities it is the free I -coloured cooperad $T_I(E[-1])$ (under the correspondence of Proposition 3.3).

The definition of P^\perp as the kernel of $T'(E[-1]) \longrightarrow T'(R')$ where $R' = T'(E[-1])(3)/R(3)[-2]$, assures that P^\perp is an I -cooperad since R is an ideal. QED

Let P be a quadratic operad. The Koszul complex of a P -algebra K is the cofree P^\perp -coalgebra on the shifted vector space $K[-1]$, with the natural differential obtained from the P -algebra structure on K in the sense of Ginzburg-Kapranov [3]. Its homology is denoted $H_*^P(K)$.

3.7 THEOREM *Let P be a quadratic I -coloured operad.*

- (i). *The I -operad P is Koszul iff $P^\perp \longrightarrow B_A(P)$ is a quasi isomorphism of I -operads.*
- (ii). *The I -operad P is Koszul iff $P^!$ is Koszul*

(iii). The homology $H_*^P(K)$ of the Koszul complex of a P -algebra K vanishes for every free P -algebra K iff P is Koszul.

PROOF The result follows directly from the Lemmas 3.4 and 3.6, and Koszul duality for operads over a semi-simple algebra as proved in Ginzburg-Kapranov [3]. QED

3.8 REMARK This article is devoted to one example of coloured Koszul duality. More examples can be found in [10]. Coloured Koszul duality is independently proved by Longoni and Tradler in preprint [6].

Koszul duality has a nice interpretation in terms of the model category of I -operads, the existence of which can be proved by the methods of Berger-Moerdijk [1]. Namely, if P is a Koszul I -operad, then $\mathcal{G}_I(P^\perp) \rightarrow P$ gives a concise cofibrant replacement for augmented operads P in this model category, where \mathcal{G}_I is the cobar construction from I -cooperads to I -operads (the dual construction to B_I in Lemma 3.4).

4 Strongly homotopy operads

An operad of non-symmetric pseudo operads

4.1 DEFINITION Define an \mathbb{N} -operad PsOpd as follows. As an \mathbb{N} -collection, $\text{PsOpd}(n, i)$ is spanned by planar rooted trees t with n vertices numbered 1 up to n , that satisfy $|\mathbf{l}_t(k)| - 1 = i(k)$ for $k = 1, \dots, n$, and $i(0) = |\mathbf{l}(t)| - 1$. Composition $s \circ_k t$ is defined by replacing vertex k in s by the planar rooted tree t (cf. Figure 1). More precisely, $s \circ_k t$ has vertices $\mathbf{v}(s) - \{k\} \cup \mathbf{v}(t)$, and edges $\mathbf{e}(s) \cup \mathbf{e}(t)$, where the elements of $\mathbf{l}_s(k)$ necessary to define the edges of s are interpreted as elements of $\mathbf{l}(t)$. This is well defined since the planar structure gives a natural isomorphism between $\mathbf{l}(t)$ and $\mathbf{l}_s(k)$.

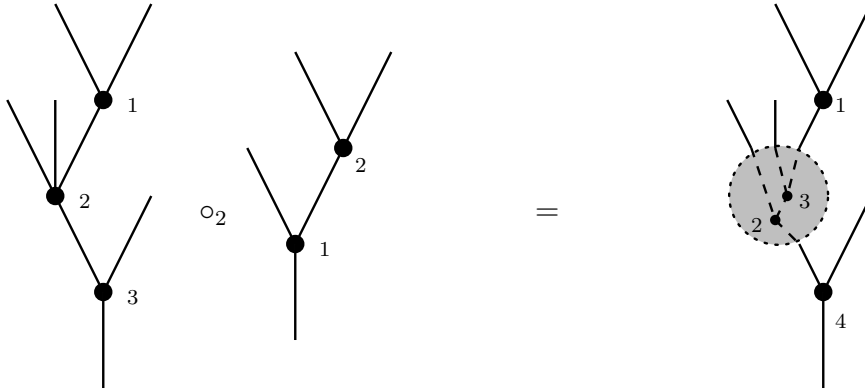


FIGURE 1: Composition \circ_2 in PsOpd : vertex 2 of the left tree is replaced by a tree with matching number of legs.

4.2 PROPOSITION The \mathbb{N} -operad PsOpd is a quadratic \mathbb{N} -operad. Algebras for PsOpd are non-symmetric pseudo operads.

PROOF Every planar rooted tree can be constructed from 2-vertex trees by compositions in PsOpd, adding one edge at a time. Denote by $(\mathbf{m} \circ_i \mathbf{n})$ the 2-vertex planar rooted tree with the root vertex having legs $\{0, \dots, m\}$, and the other vertex having legs $\{0, \dots, n\}$. The unique internal edge connects leg i of the root vertex to leg 0 of the other vertex. These generators satisfy the quadratic relations

$$(\mathbf{k} \circ_j (\mathbf{m} \circ_i \mathbf{n})) = \begin{cases} ((\mathbf{k} \circ_i \mathbf{n}) \circ_{j+n-1} \mathbf{m}) & \text{if } i < j \\ (\mathbf{k} \circ_j (\mathbf{m} \circ_{i-j+1} \mathbf{n})) & \text{if } j \leq i < j+m \\ ((\mathbf{k} \circ_{i-m-1} \mathbf{n}) \circ_j \mathbf{m}) & \text{if } j \leq i+m \end{cases} \quad (4.2)$$

These generators and relations define a quadratic \mathbb{N} -operad with free S_n -actions and non-symmetric pseudo operads as algebras, as follows from the definition. Denote this quadratic operad TE/R .

To identify the two 1-reduced operads PsOpd and TE/R it suffices to identify the free algebras on 1 generator in each colour since both \mathbb{N} -operads have free S_n -actions. Recall that the free non-symmetric pseudo operad on $A = \{A_n\}_{n \in \mathbb{N}}$ is given as $\bigoplus_t A(t)$, where the sum is over planar trees (cf. Loday [5], Appendix B). Hence the free algebras are isomorphic.

QED

Koszul duality for PsOpd

4.3 THEOREM *The \mathbb{N} -coloured operad PsOpd of non-symmetric pseudo operads is a self dual Koszul \mathbb{N} -coloured operad.*

PROOF Write $\text{PsOpd} = TE/R$ as in the proof of the previous result. We compute the Koszul dual operad $\text{PsOpd}^\dagger = T(E^*)/R^\perp$, where R^\perp is the orthogonal complement of R with respect to the pairing of TE^* and TE defined as the extension of the pairing of E^* and E twisted by a sign (cf. Ginzburg-Kapranov [3]).

The dimension of $R(3)$ is exactly half the dimension of $\text{PsOpd}(3)$, since the associativity relations divide the basis elements of $\text{PsOpd}(3)$ in pairs which satisfy a non-trivial relation. Observe that the dual relations $R^\perp(3)$ certainly are contained in the ideal generated by

$$(\mathbf{k} \circ_j (\mathbf{m} \circ_i \mathbf{n})) = \begin{cases} ((\mathbf{k} \circ_i \mathbf{n}) \circ_{j+n-1} \mathbf{m}) & \text{if } i < j \\ -(\mathbf{k} \circ_j (\mathbf{m} \circ_{i-j+1} \mathbf{n})) & \text{if } j \leq i < j+m \\ ((\mathbf{k} \circ_{i-m-1} \mathbf{n}) \circ_j \mathbf{m}) & \text{if } j \leq i+m \end{cases}$$

By a dimension argument these relations must exactly all the relations. Then a base change shows that $(\text{PsOpd})^\dagger$ is isomorphic to PsOpd. The base change is given by multiplying a basis element corresponding to a planar rooted tree t with the sign $(-1)^{c(t)}$, where $c(t)$ is the number of internal *axils* of t . That is, the number of distinct subsets $\{v, w, u\} \subset \mathbf{v}(t)$ such that two of the three vertices are direct predecessors of the third. This shows that PsOpd is self dual.

Let P be a non-symmetric pseudo operad. The PsOpd-algebra homology complex of P is as a sequence of graded vector spaces the free non-symmetric pseudo cooperad on $P[-1]$,

$$C_*^{\text{PsOpd}}(P) = F'_{\text{PsOpd}^\perp}(P) = \bigoplus_{t \text{ planar}} \bigotimes_{v \in t} P(\mathbf{l}_t(v))[-1].$$

The differential is given by contracting edges using the \circ_i -compositions in P . In other words, this complex is the non-symmetric bar construction $B_{\overline{\mathbb{Z}}}P$ (cf. Loday [5], appendix B). The Theorem follows since the homology of this complex vanishes in the case where $P = T_{\overline{\mathbb{Z}}}C$, the free non-symmetric operad on C . QED

4.4 REMARK Theorem 4.3 invites the reader to a conceptual excursion. As explained in the proof, the homology complex

$$C_*^{\text{PsOpd}}(P) = (F'_{\text{PsOpd}^\perp}(P), \partial)$$

of a non-symmetric pseudo operad P is the non-symmetric bar complex of P . This shows how bar/cobar duality for non-symmetric operads is an example of Koszul duality for the coloured operad PsOpd . The non-symmetric bar construction $B_{\overline{\mathbb{Z}}}P$ of a non-symmetric operad is nothing but the PsOpd -algebra complex of P , computing the PsOpd -algebra homology of the algebra P .

Let $P = \{P(n)\}_{n \geq 0}$ be a sequence of vector spaces. The formalism of Koszul duality defines a *strongly homotopy PsOpd-algebra* (or a *s.h. PsOpd-algebra*) is a sequence of vector spaces P , together with a square zero coderivation ∂ of the ‘cofree’ PsOpd^\perp -coalgebra on P of cohomological degree $+1$ (compare Ginzburg-Kapranov [3]).

For a planar rooted tree t , recall $P[-1](t) = \bigotimes_{v \in \mathbf{V}(t)} P_{n_v}[-1]$, where $n_v = |\mathbf{l}(v)| - 1$. A strongly homotopy PsOpd -algebra structure on P is determined by operations

$$\circ_t : P[-1](t) \longrightarrow P[-1],$$

one for every planar rooted tree t . The condition on $\partial^2 = 0$ on the differential is equivalent to a sequence of relations on these operations. For each planar rooted tree t , we obtain a relation of the form

$$\sum_{s \subset t} \pm(\circ_{t/s}) \circ (\circ_s) = 0, \quad (4.3)$$

where the sum is over (connected) planar subtrees s of t and t/s is the tree obtained from t by contracting the subtree s to a point, and the signs involved are induced by a choice of ordering on the vertices of the planar rooted trees t and s in combination with the Koszul convention. Here a connected planar subtree is a subset of vertices together with all their legs and edges such that the graph they constitute is connected. One term of the sum is illustrated in Figure 2.

Strongly homotopy operads

The s.h. PsOpd -algebras described above are not quite what we need, since these do not consider the symmetric group actions on collections.

4.5 DEFINITION Let $P = \{P(n)\}_{n \in \mathbb{N}}$ be a collection such that the vector spaces $P(n)$ form a s.h. PsOpd -algebra. Let t and t' be planar rooted trees. If $\sigma : t \longrightarrow t'$ is an isomorphism of the underlying rooted trees, then σ induces $\sigma : P(t) \longrightarrow P(t')$ through the maps of $\text{Aut}(\mathbf{l}(v))$ -modules in the tensor factors of $A(t)$, and it induced $\mathbf{l}(\sigma) : \mathbf{l}(t) \longrightarrow \mathbf{l}(t')$ and consequently

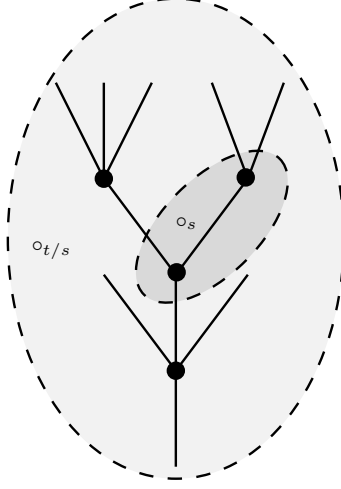


FIGURE 2: One summand of Equation (4.3): \circ_s contracts the darker part, covering subtree s of t and $\circ_{t/s}$ contracts the remaining tree.

a map of $\text{Aut}(\mathbf{l}(t))$ -modules $\mathbf{l}(\sigma) : P(\mathbf{l}(t)) \longrightarrow P(\mathbf{l}(t'))$. (Recall that the planar structure of t induces a natural identification of $P(\mathbf{l}(t))$ with $P(n)$, where $n = |\mathbf{l}(t)| - 1$.)

Call a s.h. PsOpd-algebra P *equivariant*, if for every planar rooted tree t , and every automorphism σ as above, satisfies

$$\mathbf{l}(\sigma) \circ (\circ_t) = (\circ_{t'}) \circ \sigma.$$

A *strongly homotopy operad* (or *s.h. operad*) is an equivariant strongly homotopy PsOpd-algebra.

4.6 REMARK Recall that $T'(P[-1])(n) = \lim_t P(t)$, where the limit is over the groupoid of planar rooted trees with n leafs different from the root. A differential ∂ on $F'_{\text{PsOpd}^\perp}(P)$ defined by maps $\circ_t : P(t) \longrightarrow P(\mathbf{l}(t))$ induces maps on the limit $\lim_t P(t) \longrightarrow P(\mathbf{l}(t))$ iff ∂ is equivariant (i.e. defines a strongly homotopy operad). In that case it defines a differential on $T'(P[-1])$. I use notation

$$BP = (T'(P[-1]), \partial),$$

where ∂ denotes the induced differential.

4.7 EXAMPLE The bar construction makes operads a special case of operads up to homotopy, as is suggested by the notations BP . The trees t with $|\mathbf{v}(t)| = 1$ define the internal differential, and the trees with $|\mathbf{v}(t)| = 2$ the compositions \circ_i . The \circ_t operations vanish if $|\mathbf{v}(t)| \geq 3$. The conditions on the \circ_t -operations translate into the operad axioms. Operads are exactly s.h. operads such that \circ_s vanishes if $\mathbf{v}(s) \geq 3$.

4.8 EXAMPLE Interpret operads up to homotopy as a generalisation of operads where one needs ‘higher homotopies’ that measure the failure of associativity of the \circ_i operations. I dwell a bit on this interpretation: Let P be a s.h. operad. When $|\mathbf{v}(t)| = 1$, \circ_t defines an internal differential on $P(\mathbf{l}(t))$. When $|\mathbf{v}(t)| = 2$, the operation \circ_t defines a circle- i operation

as in the definition of an (pseudo) operad. In general these operations need no longer be associative. If \circ_s does not vanish for $|\mathbf{v}(s)| = 3$, then Equation (4.3) expresses (if $|\mathbf{v}(s)| = 3$) that \circ_s serves as a homotopy for associativity as follows. Denote the internal differential by d and the two contractions of the internal edges e or e' of s by \circ_e and $\circ_{e'}$ that correspond to operad compositions. The formula

$$(\circ_e) \circ (\circ_{e'}) - (\circ_{e'}) \circ (\circ_e) = d \circ (\circ_s) + (\circ_s) \circ d$$

shows that associativity of the \circ_e compositions holds up to the homotopies \circ_s with $|\mathbf{v}(s)| = 3$. More explicitly (with the signs), for a linear tree labelled with elements p, q, r in P we have

$$\begin{aligned} & (p \circ_e q) \circ_{e'} r - p \circ_e (q \circ_{e'} r) \\ &= d(\circ_s(p, q, r)) + \circ_s(dp, q, r) + (-1)^{|p|} \circ_s(p, dq, r) + (-1)^{|p|+|q|} \circ_s(p, q, dr). \end{aligned}$$

Consequently, if P is an s.h. operad, then the cohomology H^*P with respect to the internal differential d is a graded operad. The \circ_i -compositions are induced by the operations \circ_t for trees t with 2 vertices.

Homotopy homomorphisms

In the spirit of Koszul duality, we define a *homotopy homomorphism of homotopy PsOpd[⊥]-algebras* to be a morphism of cofree PsOpd[⊥]-coalgebras compatible with the differentials. Such a morphism is a quasi isomorphism if the underlying map of vector spaces is a quasi isomorphism. Recall that by the moves of Markl [7] (we need to extend the theory to coloured operads but this is no problem) such a quasi isomorphism has a quasi inverse. A morphism of homotopy PsOpd-algebras $\varphi : A \rightsquigarrow B$ is completely determined by its restrictions

$$\varphi_t : (A[-1])(t) \longrightarrow B(\mathbf{l}(t))[-1].$$

The condition that φ is compatible with the differential can be described in terms of conditions about compatibility with the \circ_t operations:

$$\sum_{s \subset t} \pm \varphi_{t/(s)} \circ (\circ_s) = \sum_{n, s_1, \dots, s_n \subset t} \pm (\circ_{t/(s_1, \dots, s_n)}) \circ (\varphi_{s_1} \otimes \dots \otimes \varphi_{s_n}). \quad (4.4)$$

where the sum in the left hand side is over subtrees, and the sum in the right hand side for each n is over n -tuples of (connected) subtrees of t with disjoint sets of internal edges that together cover all vertices of t . The \pm is the sign induced by the Koszul convention.

4.9 DEFINITION A *homotopy homomorphism* $\varphi : A \rightsquigarrow B$ of equivariant homotopy PsOpd-algebras is *equivariant* if for any planar rooted trees t and t' and any isomorphism $\sigma : t \longrightarrow t'$ of the underlying rooted trees, the equation

$$\mathbf{l}(\sigma) \circ (\varphi_t) = \varphi_{t'} \circ \sigma$$

is satisfied. A *morphism of operads up to homotopy* $\varphi : P \rightsquigarrow Q$ is an equivariant homotopy homomorphism $\varphi : P \rightsquigarrow Q$ of equivariant homotopy PsOpd-algebras. An equivariant s.h. morphism $\varphi : P \rightsquigarrow Q$ induces a morphism of dg cooperads $\varphi : BP \longrightarrow BQ$.

Note that φ is determined by maps $\varphi_t : (P[-1])(t) \longrightarrow P[-1]$. A *homotopy quasi isomorphism* is a homotopy homomorphism such that the morphism φ_\bullet of dg collections is a isomorphism in cohomology. Here φ_\bullet stands for the restriction of φ to the 1-vertex trees.

4.10 THEOREM *Let P and Q be s.h. operads, and let $\varphi : P \rightsquigarrow Q$ be a quasi isomorphism of s.h. operads. Then there exists a quasi inverse $\psi : Q \rightsquigarrow P$ to φ .*

PROOF A homotopy quasi isomorphism of operads φ has a quasi inverse as a morphism of homotopy PsOpd-algebras (cf. Markl [7]). Let ψ denote this quasi inverse. This quasi inverse can be symmetrised as follows. Let t be a planar rooted tree with n vertices. Define

$$\psi'(t) = \frac{1}{|\text{Aut}(t)|} \sum_{\sigma \in \text{Aut}(t)} \psi_{\sigma(t)} \circ \sigma.$$

Since φ is equivariant, ψ' still is a quasi inverse to φ . Moreover, for $\tau \in \text{Aut}(t)$

$$|\text{Aut}(t)| \cdot \psi'_t \circ \tau = \sum_{\sigma \in \text{Aut}(t)} \psi_{\sigma(t)} \circ \sigma \circ \tau = \sum_{\sigma' \in \text{Aut}(t)} \psi_{\sigma' \circ \tau(t)} \circ \sigma',$$

where we use $\sigma' = \sigma \circ \tau^{-1}$ to compare the sums. Then ψ' is an equivariant quasi inverse to φ . QED

4.11 COROLLARY *Two augmented operads P and Q are quasi isomorphic iff there exists a quasi isomorphism $P \rightsquigarrow Q$ of operads up to homotopy.*

PROOF By definition P and Q are quasi isomorphic iff there exists a sequence of quasi isomorphisms of augmented operads $P \longleftarrow \cdots \longrightarrow Q$. The previous theorem can be applied to make all arrows point in the same direction if we allow s.h. maps.

On the other hand, if there exists an s.h. quasi isomorphism $P \rightsquigarrow Q$, then the bar-cobar adjunction (cf. Getzler-Jones [2]) gives a strict quasi isomorphism $\mathcal{G}(BP) \longrightarrow Q$, where $\mathcal{G}(C)$ denotes the cobar construction on a cooperad C . Moreover, there exists a natural quasi isomorphism $\mathcal{G}(BP) \longrightarrow P$. QED

5 Homotopy Algebras

Endomorphism operads

This section constructs homotopy homomorphisms between endomorphism operads, some even compatible with the identity. Well known boundary conditions turn up naturally in this context (compare Huebschmann-Kadeishvili [4]).

5.1 DEFINITION An s.h. operad is *strictly unital* if there exists an element $\text{id} \in P(1)$ that is a left and right identity with respect to the \circ_i operations where $|\mathbf{v}(t)| = 2$ and such that the other compositions \circ_i vanish when applied to id in one coordinate. A homotopy homomorphism φ of two strictly unital operads up to homotopy is *strictly unital* if the underlying morphism φ_\bullet of collections preserves the identity, and if for $|\mathbf{v}(t)| > 1$, the map $\varphi(t)$ vanishes when applied to id in one coordinate.

Let V and W be dg vector spaces. V is a *strict deformation retract* of W if there exist an inclusion $i : V \longrightarrow W$ and a retraction $r : W \longrightarrow V$ such that both i and r are dg maps, $r \circ i = \text{id}_V$, and there exists a chain homotopy H between $i \circ r$ and id_W , satisfying the boundary conditions $H \circ i = 0$, $r \circ H = 0$, and $H \circ H = 0$.

5.2 THEOREM *Let V and W be dg vector spaces. Let $i : V \longrightarrow W$ and $r : W \longrightarrow V$ be dg linear maps, and $H : W \longrightarrow W[1]$ a chain homotopy between $i \circ r$ and id_W .*

- (i). *There exists a (non-unital) homotopy homomorphism $\varphi : \text{End}_W \rightsquigarrow \text{End}_V$ (defined by the Formula (5.5) below).*
- (ii). *If i and r are quasi isomorphisms, then φ is a quasi isomorphism.*
- (iii). *If the data above make V a strict deformation retract of W , then φ is strictly unital.*

PROOF The map φ_\bullet corresponding to 1 vertex trees is $f \longmapsto r \circ f \circ i^{\otimes n}$ for $f \in \text{End}_V(n)$. This proves the second part of the Theorem. Define an alternative composition $\hat{\gamma}$ on End_V by $f \hat{\gamma}_i g = f \circ_1 H' \circ_i g$, where $H'(x) = (-1)^{|x|} H(x)$. This composition makes End_V a pseudo operad. For a planar rooted tree t , the map

$$\varphi(t) = \varphi_\bullet \circ \hat{\gamma}_t, \quad (5.5)$$

where $\hat{\gamma}_t : \text{End}_W(t) \longrightarrow \text{End}_W(\mathbf{1}(t))$ is the composition based on $\hat{\gamma}$. This is visualised in Figure 3.

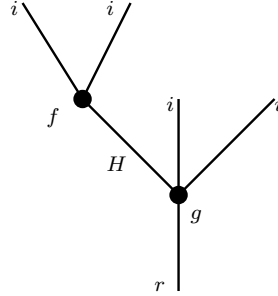


FIGURE 3: The map $\varphi(t)(f, g, h) = r \circ g \circ ((H' \circ f \circ i^{\otimes 2}), i^{\otimes 2})$, represented by a tree with labelled internal and external edges.

It remains to check Formula (4.4). For a fixed tree t this reduces to

$$\sum_{e \in \mathbf{e}(t)} (\circ_e) \circ (\varphi(t^e) \otimes \varphi(t_e)) + d \circ \varphi(t) = \sum_{e \in \mathbf{e}(t)} \varphi(t/e) \circ (\circ_e) + \varphi(t) \circ d. \quad (5.6)$$

The argument that this hold is the following. Since r and i commute with the differential d , and the internal differentials act as a derivation with respect to composition of multi-linear maps, the formula follows from the equalities $d \circ H + H \circ d = \text{id} - i \circ r$ applied to the summand for each edge e . This shows part (i).

Assume the conditions of (iii). To assure that φ_\bullet preserves the identity, use $r \circ i = \text{id}_V$. The conditions on compositions with H assure that higher operations applied to the identity vanish. QED

5.3 REMARK Let us take a closer look at the proof above. Since the cancellation of terms is local with respect to the geometry of the tree t (i.e. cancellation per edge), it suffices to check the signs for a tree with one edge as in Figure 3. Let us do the calculation with the signs for this tree. We leave out the pre-composition with i and post composition with r in the final terms. The usual degree of f is denoted by $|f|$. The left hand side of Equation (5.6) reads

$$g \circ_k i \circ r \circ f + d \circ g \circ_k H \circ f + (-1)^{|f|+|g|+1} g \circ_k H \circ f \circ d.$$

The right hand side equals

$$\begin{aligned} & g \circ_k f + (-1)^{|g|} g \circ d \circ_k H \circ f + (-1)^{|g|+1} g \circ_k H \circ d \circ f \\ & + d \circ g \circ_k H \circ f + (-1)^{|f|+|g|+1} g \circ_k H \circ f \circ d. \end{aligned}$$

To obtain the signs, note that we have a sign from moving d in, and note that these signs are with respect to the shifted grading on End_V and End_W , while the sign in $d(f) = d \circ f + (-1)^{|f|} f \circ d$ is with respect to the usual grading. The signs are correct if we replace H by $H'(x) = (-1)^{|x|} H(x)$.

Homotopy Q -algebras

I already discussed homotopy algebras for Koszul operads. This section discusses the more general approach to homotopy algebras. It shows how operads up to homotopy can be used to give a different interpretation of the usual definition.

5.4 DEFINITION Let Q be an augmented operad. A *homotopy Q -algebra* structure on a dg vector space V is a homotopy homomorphism $Q \rightsquigarrow \text{End}_V$.

Recall that this induces a map of cooperads $BQ \longrightarrow B\text{End}_V$. To such a morphism corresponds by the bar/cobar adjunction a morphism of operads $\mathcal{H}(BQ) \longrightarrow \text{End}_V$, where $\mathcal{H} : \text{Coopd} \longrightarrow \text{Opd}$ is the cobar construction as in the proof of Corollary 4.11. Moreover, $\mathcal{H}B(Q)$ is a cofibrant replacement of Q in the model category of operads (cf. the proof of Corollary 4.11 for the notation). This explains the terminology.

5.5 PROPOSITION *Let Q be an augmented operad.*

- (i). *let W be a homotopy Q -algebra, and W a dg vector space. If $i : V \longrightarrow W$, $r : W \longrightarrow V$ are quasi isomorphisms, and $H : r \circ i \sim \text{id}_W$, then V has the structure of a homotopy Q -algebra such that the induced maps in cohomology $H(r)$ and $H(i)$ are isomorphisms of Q -algebras.*
- (ii). *Let V be a homotopy Q -algebra, and let W be a dg vector space. If $i : V \longrightarrow W$, $r : W \longrightarrow V$ are quasi isomorphisms, and $H : r \circ i \sim \text{id}_W$, then W has the structure of a homotopy Q -algebra such that $H(r)$ and $H(i)$ are isomorphisms of Q -algebras.*

PROOF Suppose that W is a homotopy Q -algebra. Recall that we constructed from the data in the Theorem a quasi isomorphism $\text{End}_W \rightsquigarrow \text{End}_V$ in Theorem 5.2. The composition

$$BQ \longrightarrow B\text{End}_W \longrightarrow B\text{End}_V$$

defines the desired homotopy homomorphism $Q \rightsquigarrow \text{End}_V$, where the map $BQ \rightarrow B\text{End}_W$ is the map defined by the homotopy Q -algebra structure on W , which proves (i).

Suppose that V is a homotopy Q -algebra. The quasi isomorphism $\text{End}_W \rightsquigarrow \text{End}_V$ has a quasi inverse (by Theorem 4.10), and thus we can construct the composition $BQ \rightarrow B\text{End}_V \rightarrow B\text{End}_W$, which defines a homotopy Q -algebra structure on W . QED

5.6 REMARK Observe that all the results can be generalised to coloured operads: An *strongly homotopy I -operad* P is an I -collection P together with a differential ∂ on the ‘cofree’ pseudo I -cooperad $T'_I(P[-1])$. If we denote $B_IP = (T'_I(P[-1]), \partial)$, we can define a homotopy homomorphism $P \rightsquigarrow Q$ of s.h. I -operads as a morphism of I -cooperads $B_IP \rightarrow B_IQ$. Notably, for sequences of vector spaces $V = \{V_\alpha\}_{\alpha \in I}$ and $W = \{W_\alpha\}_{\alpha \in I}$ such that for each W_α and V_α we have i_α, r_α and H_α as in the second part of the Proposition above, we can find a quasi isomorphism $\text{End}_W \rightsquigarrow \text{End}_V$, which yields the analogue of Proposition 5.5 for algebras over I -operads.

Example: configuration spaces

Let D_2 be the *operad of little disks*. That is, D_2 is the topological operad such that $D_2(n)$ is the space of ordered n -tuples of disjoint embedding of the unit disk D_2 in D_2 that preserve horizontal and vertical directions. The operations \circ_k are defined by compositions of embeddings.

Let $F(n)$ denote the *configuration space* of n distinct ordered points in the open unit disk in \mathbb{R}^2 . Thus $F(n)$ is the n -fold product of the unit disk with the (sub)diagonals cut out. Consider $F = \{F(n)\}_{n \geq 1}$ as a collection with respect to permutation of the order of the points.

For a topological space X , denote by $S_*(X)$ the singular k -chain complex on X with coefficients in k .

5.7 THEOREM *The singular \mathbb{Q} -chains $S_*(F)$ on configuration spaces form an operad up to homotopy quasi isomorphic (in the sense of Proposition 5.5) to the operad $S_*(D_2)$ of singular \mathbb{Q} -chains on the little disks operad.*

PROOF We first sketch the line of argument. We construct an S_n -equivariant homotopy between the little disks and the configuration spaces. It then follows that $S_*(F)$ is a homotopy algebra for PsOpd homotopy equivalent to $S_*(D_2)$. Since the homotopy algebra $S_*(F)$ is equivariant, $S_*(F)$ is an s.h. operad. This is based on the observation in Remark 5.6 that the all results go through for coloured operads.

Then there exists an inclusion $i : F(n) \rightarrow D_2(n)$ and a retraction $r : D_2(n) \rightarrow F(n)$ such that $\text{id} \sim i \circ r$ by a homotopy H , and $r \circ i = \text{id}$. Consider points in $D_2(n)$ as given by a n -tuple (x_1, \dots, x_n) of points in the interior of D_2 and an n -tuple (r_1, \dots, r_n) of radii, and a point in $F(n)$ by a n -tuple (x_1, \dots, x_n) of points in the interior of D_2 . One might take the retraction r by defining all radii in $r(x_1, \dots, x_n)$ equal to

$$\frac{1}{3}(\min(\{|x_i - x_j| \mid (i \neq j)\} \cup \{1 - |x_i|\})).$$

(The map r is not smooth but only continuous.) A homotopy H between $i \circ r$ and the identity is readily defined by drawing a tube of configurations

with the two configurations at the boundary disks, connection the little disks by straight lines. (cf. Figure 4).

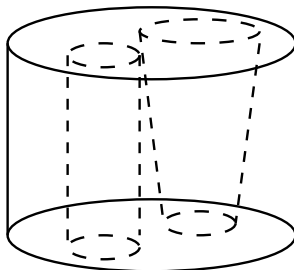


FIGURE 4: Construction of the homotopy H . The tubes do not intersect since the centres of the disks are fixed.

The homotopy H induces a chain homotopy between $S_*(i) \circ S_*(r)$ and the identity. Theorem 5.2 then shows that there exists a homotopy homomorphism of \mathbb{N} -operads $\text{End}_{S_*(D_2)} \longrightarrow \text{End}_{S_*(F)}$. By composition with the morphism $\text{PsOpd} \longrightarrow \text{End}_{S_*(D_2)}$, the \mathbb{N} -collection $S_*(F)$ is a homotopy algebra for the \mathbb{N} -operad PsOpd (cf. Proposition 5.5). Both i and r (and thus $S_*(i)$ and $S_*(r)$) are compatible with the symmetric group actions on $D_2(n)$ and $F(n)$. Consequently, this makes the singular chains $S_*(F)$ an equivariant homotopy PsOpd -algebra, and thus an operad up to homotopy. QED

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